South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 1 (A) (2020), pp. 55-60

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

ESTIMATES OF INITIAL COEFFICIENTS FOR FUNCTIONS IN NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED USING GENERALIZED DIFFERENTIAL OPERATOR

Jenifer Arulmani

Department of Mathematics,
Presidency College(Autonomous),
Chennai - 600005, Tamil Nadu, INDIA

E-mail : jeniferarulmani06@gmail.com

(**Received:** *Mar. 03, 2020* **Accepted:** *April. 26, 2020* **Published:** *Apr. 30, 2020)*

Abstract: We defined a new subclass of the function class $\sum$ in the unit disk associated with a generalized differential operator. We estimated some coefficient bounds for the functions in the defined new subclass. As special cases, well-known results were obtained by varying parameters in the main results.

Keywords and Phrases: Coefficient bounds, bi-univalent, subclasses, differential operator.

2010 Mathematics Subject Classification: Primary 30C45, secondary 30C50.

1. Introduction and Preliminaries

We denote by A the class of regular functions defined in the open unit disk $\Delta = \{z : |z| < 1\}$ with the normalization conditions $f(0) = f'(0) - 1 = 0$ and the Taylor series expansion,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

We know that "The range of every function of the class S contains a disk $\{w : |w| < 1/4\}$", by the Koebe-one quater theorem [3] (Theorem 2.3 pg. 31). Hence there exists inverse f^{-1} for every function $f \in S$, defined by $f^{-1}(f(z)) = z, (z \in \Delta)$ and

$$f(f^{-1}(w)) = w, (|w| < r_0(f) : r_0(f) \geq 1/4).$$

Where the inverse of f is given by,

$$\begin{aligned} f^{-1}(w) &= w - a_2w^2 + (2a_2^2w^2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4 + ... \\ &=: g(w) \end{aligned}$$

A function $f \in A$ is said to be bi-univalent if both f and f^{-1}(its inverse) are univalent in Δ. We denote by $\sum$ the class of bi-univalent and analytic functions in Δ of the form (1). Using the binomial series,

$$(1-\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j, m \in \mathbb{N} = 1, 2, ... \textit{ and } j \in \mathbb{N}_0 = 0, 1, 2, ...$$

Frasin [4] defined the following differential operator for function $f \in A$,

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1_{m,\lambda} f(z) &= (1-\lambda)^m f(z) + (1-(1-\lambda)^m) z f'(z) \\ &= D_{m,\lambda} f(z) \ , \ (\lambda > 0; m \in \mathbb{N}). \end{aligned}$$

In general,

$$\begin{aligned} D^n_{m,\lambda} f(z) &= D_{m,\lambda}(D^{n-1}_{m,\lambda} f(z)) \ , \ n \in \mathbb{N}_0 \\ &= z + \sum_{k=2}^{\infty} [1 + (k-1) c_j^m(\lambda)]^n a_k z^k \end{aligned}$$

where, $c_j^m(\lambda) = \sum\limits_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j$.

Remark 1.1.

1. *For $m = 1$, we get the Al-oboudi differential operator, $D^n_{1,\lambda}$ [1].*
2. *For $m = \lambda = 1$, we get the Salagean differential operator, D^n [7].*

We consider $\mathbb{P}$ to be the class of Caratheodary functions. i.e., for $p(z) \in \mathbb{P}$, $\Re\{p(z)\} > 0$, $p(z)$ is analytic in Δ and have the series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ , \ z \in \Delta.$$

Lemma 1.2. *[3] If $p(z) \in \mathbb{P}$, then $|p_n| \leq 2$ for each $n = 1, 2, ...$*

2. The function class $S_{\sum}^{n,\delta}(m,\lambda,\alpha)$

Definition 2.1. *A function $f(z)$ in $\sum$ of the form (1) is said to be in the class $S_{\sum}^{n,\delta}(m,\lambda,\alpha)$ where $(0<\alpha\le 1; 0\le\delta<1; m\in\mathbb{N}; n\in\mathbb{N}_0, \lambda>0)$ if the following conditions are satisfied*

$$f\in\sum,\ |arg(\frac{D_{m,\lambda}^{n+1}f(z)}{(1-\delta)D_{m,\lambda}^{n}f(z)+\delta D_{m,\lambda}^{n+1}f(z)})|<\frac{\alpha\pi}{2}\quad (z\in\Delta)$$

and

$$f\in\sum,\ |arg(\frac{D_{m,\lambda}^{n+1}g(w)}{(1-\delta)D_{m,\lambda}^{n}g(w)+\delta D_{m,\lambda}^{n+1}g(w)})|<\frac{\alpha\pi}{2}\quad (w\in\Delta).$$

Remark 2.2.

1. *When $m=\lambda=1$ $S_{\sum}^{n,\delta}(1,1,\alpha)=S_{\sum}^{n,\delta}(\alpha)$, the class introduced and studied by Jothibasu [5].*
2. *When $m=\lambda=1$ and $n=0$, $S_{\sum}^{0,\delta}(1,1,\alpha)=S_{\sum}^{\delta}(\alpha)$, the class studied by Murugasundaramoorthy [6].*
3. *When $m=\lambda=1$ and $\delta=n=0$, $S_{\sum}^{0,0}(1,1,\alpha)=SS_{\sum}^{*}(\alpha)$, the class of strongly bi-starlike function studied by Brannan et al., [2].*
4. *When $m=\lambda=n=1$ and $\delta=0$, $S_{\sum}^{1,0}(1,1,\alpha)=SK_{\sum}(\alpha)$, the class of strongly bi-convex function studied by Brannan. et al., [2].*

3. Coefficient bounds for the class $S_{\sum}^{n,\delta}(m,\lambda,\alpha)$

Theorem 3.1. *For functions $f(z)$ given by (1) is in the class $S_{\sum}^{n,\delta}(m,\lambda,\alpha)$, then*

$$|a_2|\le\frac{2\alpha}{\sqrt{4\alpha c_j^m(\lambda)(1+2c_j^m(\lambda))^n(1-\delta)+c_j^m(\lambda)[\chi(1+c_j^m(\lambda))^{2n}]}}$$

and

$$|a_3|\le\frac{\alpha}{c_j^m(\lambda)(1+2c_j^m(\lambda))^n(1-\delta)}+\frac{4\alpha^2}{c_j^m(\lambda)(1-\delta)^2(1+c_j^m(\lambda))^{2n}}$$

where $\chi=2\alpha(1+c_j^m(\lambda)\delta)(\delta-1)-(\alpha-1)(1-\delta)^2c_j^m(\lambda)$

Proof. From the definition 2.1, it follows that

$$\frac{D_{m,\lambda}^{n+1}f(z)}{(1-\delta)D_{m,\lambda}^{n}f(z)+\delta D_{m,\lambda}^{n+1}f(z)}=[p(z)]^\alpha$$

$$\frac{D^{n+1}_{m,\lambda}g(w)}{(1-\delta)D^{n}_{m,\lambda}g(w)+\delta D^{n+1}_{m,\lambda}g(w)} = [q(w)]^{\alpha}$$

where $p(z)$ and $q(z)$ are in $\mathbb{P}$ and have the form

$$p(z) = 1 + \sum_{k=0}^{\infty} p_k z^k = 1 + p_1 z + p_2 z^2 + ...$$

$$q(w) = 1 + \sum_{k=0}^{\infty} q_k w^k = 1 + q_1 w + q_2 w^2 + ...$$

Equating the coefficients of z, z^2, w and w^2 we have

$$c_j^m(\lambda)(1-\delta)(1+c_j^m(\lambda))^n a_2 = \alpha p_1 \tag{2}$$

$$c_j^m(\lambda)(1+c_j^m(\lambda)\delta)(\delta-1)(1+c_j^m(\lambda))^{2n}a_2^2 + 2c_j^m(\lambda)(1+2c_j^m(\lambda))^n(1-\delta)a_3$$

$$= (\frac{(\alpha-1)}{2}p_1^2 + p_2)\alpha \tag{3}$$

$$-c_j^m(\lambda)(1-\delta)(1+c_j^m(\lambda))^n a_2 = \alpha p_1 \tag{4}$$

$$c_j^m(\lambda)(1+c_j^m(\lambda)\delta)(\delta-1)(1+c_j^m(\lambda))^{2n}a_2^2 + 2c_j^m(\lambda)(1+2c_j^m(\lambda))^n(1-\delta)(2a_2^2-a_3)$$

$$= (\frac{(\alpha-1)}{2}p_1^2 + p_2)\alpha \tag{5}$$

from (2) and (4), we have

$$p_1 = -q_1 \tag{6}$$

and by squaring and adding (2) and (4), we have

$$2(c_j^m(\lambda))^2(1-\delta)^2(1+c_j^m(\lambda)))^{2n}a_2^2 = \alpha^2(p_1^2+q_1^2) \tag{7}$$

$$(p_1^2+q_1^2) = \frac{2(c_j^m(\lambda))^2(1-\delta)^2(1+c_j^m(\lambda)))^{2n}}{\alpha^2}. \tag{8}$$

Adding (3) and (5), we have

$$(2c_j^m(\lambda)(1+c_j^m(\lambda)\delta)(\delta-1)(1+c_j^m(\lambda))^{2n} + 4c_j^m(\lambda)(1+2c_j^m(\lambda))^n(1-\delta))a_2^2$$

$$= \frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)+\alpha(p_2+q_2). \tag{9}$$

Using (8) and (9) we have

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{4\alpha c_j^m(\lambda)(1+2c_j^m(\lambda))(1-\delta)+\chi c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}}$$

where $\chi = 2\alpha(1+c_j^m(\lambda)\delta)(\delta-1)-(\alpha-1)(1-\delta)^2 c_j^m(\lambda)$. Now using Lemma 1.1 and triangular inequality we arrive at the desired estimate for $|a_2|$.
To find the bound for $|a_3|$ we subtract (5) from (3), we get

$$4c_j^m(\lambda)(1+2c_j^m(\lambda))^n(1-\delta)(a_3-a_2^2) = \frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2)+(p_2-q_2)\alpha.$$

Using (6) in the above equation we have

$$a_3 = \frac{\alpha(p_2-q_2)}{4c_j^m(\lambda)(1+2c_j^m(\lambda))^n(1-\delta)}+a_2^2.$$

Using (7), Lemma 1.1 and triangular inequality we get the bound for $|a_3|$.
For $m=1$, we get the following corollary

Corollary 3.2. *For functions $f(z)$ in the class $S_{\Sigma}^{n,\delta}(1,\lambda,\alpha)$, then*

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha\lambda(1+2\lambda)^n(1-\delta)+\lambda[\zeta(1+\lambda)^{2n}]}}$$

and

$$|a_3| \le \frac{\alpha}{\lambda(1+2\lambda)^n(1-\delta)}+\frac{4\alpha^2}{\lambda(1-\delta)^2(1+\lambda)^{2n}}$$

where $\zeta = 2\alpha(1+\lambda\delta)(\delta-1)-(\alpha-1)(1-\delta)^2\lambda$.
For $m=\lambda=1$, we get the result of Jothibasu [5],

Corollary 3.3. *For functions $f(z)$ in the class $S_{\Sigma}^{n,\delta}(\alpha)$, then*

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha(1-\delta)3^n+[2\alpha(\delta^2-1)-(\alpha-1)(1-\delta)^2]2^{2n}}}$$

and

$$|a_3| \le \frac{\alpha}{3^n(1-\delta)}+\frac{4\alpha^2}{(1-\delta)^2 2^{2n}}.$$

Remark 3.4.

1. *For $m = \lambda = 1$ and $n = \delta = 0$, we get the estimates for the bi-starlike functions $SS^*_{\sum}(\alpha)$ results of Brannan et al in [2].*

2. *For $m = \lambda = n = 1$ and $\delta = 0$, we get the estimates for the bi-convex functions $SK^*_{\sum}(\alpha)$ results of Brannan et al in [2].*

References

[1] Al-oboudi, F. M., On univalent functions defined by a generalized Salagean operator, International Journal of Mathematics and Mathematical Sciences, 27(2004), pp.1429-1436.

[2] Brannan, D. A. and Taha, T. S., On some classes of bi-univalent functions, Studia Universitatis Babes-Bolyai, Mathematica, 31(2)(1986), pp. 70-77.

[3] P. L. Duren, Univalent functions, Springers, 1983.

[4] B. A. Frasin, A new differential operator of analytic functions involving binomial series, Bol. Soc. Paran. Mat, V-38, No-5(2020), pp. 205-213.

[5] Jothibasu, J., Certain subclasses of bi-univalent functions defined using Salagean operator, Elect. J. Math. ANal. Appl, 3(1)(2015), pp. 155-157.

[6] Murugasundaramoorthy, G. Selvaraj, C. and Babu, O. S., Coefficient estimates for Pascu-type subclasses of bi-univalent functions based on subordination, International Journal of Nonlinear Science, V. 19, No. 1 (2015), pp. 47-52.

[7] Salagean, G. S., Subclasses of univalent functions, Lectures notes in Mathematics, Springers, 1013(1983).